

# Universality for conditional measures of the sine point process

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## Abstract

The sine process is a rigid point process on the real line, which means that for almost all configurations  $X$ , the number of points in an interval  $I = [-R, R]$  is determined by the points of  $X$  outside of  $I$ . In addition, the points in  $I$  are an orthogonal polynomial ensemble on  $I$  with a weight function that is determined by the points in  $X \setminus I$ . We prove a universality result that in particular implies that the correlation kernel of the orthogonal polynomial ensemble tends to the sine kernel as the length  $|I| = 2R$  tends to infinity, thereby answering a question posed by A.I. Bufetov.

## 1 Introduction and statement of results

### 1.1 Introduction

The aim of this paper is to prove a universality result for determinantal point processes on the real line. The result is that certain correlation kernels, associated with orthogonal polynomial ensembles, tend to the universal sine kernel

$$\frac{\sin \pi(x - y)}{\pi(x - y)}, \quad (1.1)$$

see Theorems 1.3 and 1.4 for the precise statements. Such limiting results of orthogonal polynomial ensembles are well-known in the theory of random

matrices, see e.g. [10, 18, 21]. The kernel (1.1) is the typical limiting kernel for bulk eigenvalue correlations of random Hermitian matrices, also beyond invariant ensembles [13]. The sine kernel is the correlation kernel for a translation invariant determinantal point process  $\mathbb{P}_{\text{sin}}$  on  $\mathbb{R}$ , normalized in such a way that neighboring points have distance 1 on average.

The problem we address is motivated by recent studies on the sine process and other determinantal processes by Bufetov [5]. The starting point of [5] is the quasi-invariance of point processes, originating from the work of Olshanski [22], combined with the number rigidity of the sine process in the sense of Ghosh and Peres [14, 16]. The number rigidity (or simply rigidity) of a locally finite point process  $\mathbb{P}$  on the real line means the following. For any compact interval  $I$ , and for  $\mathbb{P}$ -almost all point configurations  $X$ , it is true that  $X \setminus I$  determines the number of points in  $X \cap I$  almost surely. Thus, by observing the points outside of  $I$ , one can deduce the number of points inside  $I$ , with probability one. Number rigidity is a very remarkable property, which makes the sine process very different from a Poisson point process. Indeed, in a Poisson point process, the points in  $I$  and those in  $I^c = \mathbb{R} \setminus I$  are independent and so from knowing the points in  $I^c$  one gains no extra information about the points in  $I$ . For more examples of rigid point processes, see [4, 7, 8, 15].

Number rigidity can be expressed in terms of conditional measures. Let  $I$  again be a compact interval, and let  $Y$  be a locally finite configuration of points in  $I^c$ . The conditional measure  $\mathbb{P}(\cdot \mid Y; I^c)$  of a point process  $\mathbb{P}$  is a new point process that is supported on configurations  $X$  with  $X \cap I^c = Y$ . Informally, it is the point process obtained from  $\mathbb{P}$  by conditioning that the points outside of  $I$  coincide with  $Y$ . See [5] or [23, Chapter 5, Section 8] for a precise description. For a rigid point process  $\mathbb{P}$ , and for  $\mathbb{P}$ -almost all  $X$  the conditional measure  $\mathbb{P}(\cdot \mid X \setminus I; I^c)$  is identified with a point process on  $I$  with exactly  $N = \#(X \cap I)$  points.

For a large class of rigid determinantal point processes  $\mathbb{P}$  on  $\mathbb{R}$ , including the sine process, Bufetov [5, Theorems 1.1 and 1.4] showed that, for  $\mathbb{P}$  almost all  $X$ , the conditional measure  $\mathbb{P}(\cdot \mid X \setminus I; I^c)$ , when considered as a point process on  $I$ , has a joint density on  $I^N$  of the form

$$\frac{1}{Z_{I,X}} \prod_{j < k} (t_k - t_j)^2 \prod_{j=1}^N \rho_{I,X}(t_j), \quad N = \#(X \cap I), \quad (1.2)$$

for certain functions  $\rho_{I,X}$  that in addition to  $I$  and  $X$  also depend on the point process. The arguments leading to (1.2) are based on quasi-invariance properties [3, 22] of the point processes. For the sine process these functions

are such that, see [5, Corollary 1.5]

$$\frac{\rho_{I,X}(t_1)}{\rho_{I,X}(t_2)} = \lim_{R \rightarrow \infty} \prod_{p \in X \setminus I, |p| \leq R} \left( \frac{t_1 - p}{t_2 - p} \right)^2, \quad t_1, t_2 \in I.$$

Hence, one might take

$$\rho_{I,X}(t) = \prod_{p \in X \setminus I} \left( 1 - \frac{t}{p} \right)^2, \quad t \in I, \quad (1.3)$$

with a product that converges in principal value.

A joint density of the form (1.2) is determinantal with a correlation kernel that is built out of the orthogonal polynomials for the weight  $\rho_{I,X}$  on  $I$ . Namely, if  $(\varphi_j)_{j=0}^\infty$  is the sequence of orthonormal polynomials, i.e.,  $\deg \varphi_j = j$  and

$$\int_I \varphi_j(t) \varphi_k(t) \rho_{I,X}(t) dt = \delta_{j,k},$$

then the kernel is

$$K_{I,X}(x, y) = \sqrt{\rho_{I,X}(x) \rho_{I,X}(y)} \sum_{j=0}^{N-1} \varphi_j(x) \varphi_j(y). \quad (1.4)$$

If we now take  $I$  centered at the origin, it will be reasonable to expect that the effect of conditioning on  $X \setminus I$  becomes less important as the length of  $I$  tends to infinity. In the limit  $|I| \rightarrow \infty$  we expect to recover the kernel of the point process  $\mathbb{P}$ . For the sine process this was stated as an open problem by A.I. Bufetov [6] to one of us.

**Problem 1.1 (Bufetov)** *For a locally finite configuration  $X$  on  $\mathbb{R}$ , and for an interval  $I$ , let  $K_{I,X}$  be the orthogonal polynomial kernel (1.4) associated with the weight function (1.3). Is it then true that for  $\mathbb{P}_{\text{sin}}$ -almost all point configurations  $X$  we have that*

$$K_{I,X}(x, y) \rightarrow \frac{\sin \pi(x - y)}{\pi(x - y)}$$

as  $I = [-R, R]$  and  $|I| \rightarrow \infty$ ?

We are going to answer Problem 1.1 in the affirmative in a more general, deterministic setting, that we will explain next.

## 1.2 Notation and statement of main results

Our results deal with a fixed, deterministic set of points  $X = \{p_n \mid n \in \mathbb{Z}\}$ . Our assumptions on  $X$  are

**Assumption 1.2** (a)  $(p_n)_{n \in \mathbb{Z}}$  is a strictly increasing doubly infinite sequence, indexed so that

$$\cdots < p_{-2} < p_{-1} < 0 \leq p_0 < p_1 < \cdots \quad (1.5)$$

(b) the series  $\sum p_n^{-1}$  converges in principal value, that is,

$$\lim_{S \rightarrow \infty} \sum_{0 < |p_n| < S} \frac{1}{p_n} \quad \text{exists,} \quad (1.6)$$

(c) and

$$\lim_{n \rightarrow \pm\infty} \frac{p_n}{n} = 1. \quad (1.7)$$

The convergence in principal value (1.6) is then (in our situation, assuming (1.5) and (1.7)) the same as saying that

$$\sum_{n=1}^{\infty} \left( \frac{1}{p_n} + \frac{1}{p_{-n}} \right) \quad \text{converges.} \quad (1.8)$$

The assumptions are satisfied for a typical configuration  $X$  from the sine process, since the sine process is a simple point process, and if  $X = \{p_n \mid n \in \mathbb{Z}\}$  with  $(p_n)$  as in (1.5) then it is known that

$$p_n = n + O(|n|^{1/2} \log^2 |n|) \quad \text{as } n \rightarrow \pm\infty \quad (1.9)$$

holds  $\mathbb{P}_{\text{sin}}$ -almost surely. It is easy to show that (1.5) and (1.9) imply (1.6) and (1.7).

For a non-negative weight function  $w$  on  $\mathbb{R}$  such that  $\int |t|^n w(t) dt < \infty$  for all  $n \geq 0$ , we denote by  $(\varphi_j(\cdot; w))_{j=0}^{\infty}$  the sequence of orthonormal polynomials with respect to  $w$ . That is,  $\varphi_j(\cdot; w)$  is a polynomial of degree  $j$  with positive leading coefficient, and

$$\int \varphi_j(t; w) \varphi_k(t; w) w(t) dt = \delta_{jk}.$$

The kernel functions associated to  $w$  are defined by

$$K_N(x, y; w) = \sqrt{w(x)w(y)} \sum_{j=0}^{N-1} \varphi_j(x; w) \varphi_j(y; w), \quad N \geq 1. \quad (1.10)$$

In the proof of Theorem 1.4 we will also consider the polynomial kernels

$$\widehat{K}_N(x, y; w) = \sum_{j=0}^{N-1} \varphi_j(x; w) \varphi_j(y; w), \quad (1.11)$$

that do not include the square root of the weight factors.

With  $X$  being fixed and  $I = [-R, R]$ , the weight (1.3) is equal to

$$\begin{aligned} \rho_R(t) &= \prod_{|p_n| > R} \left(1 - \frac{t}{p_n}\right)^2 \\ &= \lim_{S \rightarrow \infty} \prod_{R < |p_n| < S} \left(1 - \frac{t}{p_n}\right)^2. \end{aligned} \quad (1.12)$$

The limit exists because of the assumption (1.6). Because of (1.7) we have  $N(R)/R \rightarrow 2$  as  $R \rightarrow \infty$ . The main result of this paper is the following universality theorem, that also answers the Problem 1.1 of Bufetov.

**Theorem 1.3** *Let  $(p_n)_{n \in \mathbb{Z}}$  be a doubly infinite sequence satisfying Assumption 1.2. Let*

$$N(R) := \#\{p_n \mid |p_n| \leq R\}. \quad (1.13)$$

*Then we have*

$$\lim_{R \rightarrow \infty} K_{N(R)}(x, y; \rho_R) = \frac{\sin \pi(x - y)}{\pi(x - y)}$$

*uniformly for  $x$  and  $y$  in compact subsets of the real line.*

To prove Theorem 1.3 we are going to rescale the weights to the interval  $[-1, 1]$ . We consider the new weights

$$w_R(t) := \rho_R(Rt), \quad (1.14)$$

so that  $\sqrt{R}\varphi_j(t; \rho_R) = \varphi_j(t/R; w_R)$ , and consequently,

$$K_N(x, y; \rho_R) = \frac{1}{R} K_N\left(\frac{x}{R}, \frac{y}{R}; w_R\right).$$

Since  $N(R)/R \rightarrow 2$  as  $R \rightarrow \infty$ , Theorem 1.3 will follow as a corollary of the following result.

**Theorem 1.4** *Let  $(p_n)_{n \in \mathbb{Z}}$  be a doubly infinite sequence satisfying Assumption 1.2. Let  $N$  be an integer depending on  $R$  in such a way that*

$$\lim_{R \rightarrow \infty} \frac{N}{R} = 2. \quad (1.15)$$

Then we have

$$\lim_{R \rightarrow \infty} \frac{2}{N} K_N \left( \frac{2x}{N}, \frac{2y}{N}; w_R \right) = \frac{\sin \pi(x-y)}{\pi(x-y)} \quad (1.16)$$

uniformly for  $x$  and  $y$  in compact subsets of the real line.

### 1.3 Outline of the proof

A universality limit such as (1.16) is well-known for varying weights of the form  $e^{-NV(t)}$ , see [10], where  $V$  is real analytic on  $\mathbb{R}$ . Suppose that the equilibrium measure  $\mu_V$  corresponding to the external field  $V$  has a density  $\psi_V$  (see [9, 24] for main references on equilibrium measures in external fields). If  $\psi_V(0) > 0$ , then the universality says that

$$\lim_{N \rightarrow \infty} \frac{1}{\psi_V(0)N} K_N \left( \frac{x}{\psi_V(0)N}, \frac{y}{\psi_V(0)N}; e^{-NV} \right) = \frac{\sin \pi(x-y)}{\pi(x-y)}. \quad (1.17)$$

The external field  $V$  that plays a role in this paper is

$$V(t) = (1+t) \log(1+t) + (1-t) \log(1-t), \quad t \in [-1, 1]. \quad (1.18)$$

It is easy to see by direct computation that

$$V(t) = \int_{-1}^1 \log |t-s| ds, \quad \text{for } t \in [-1, 1],$$

which means that the equilibrium measure for  $V$  has the constant density

$$\psi_V(t) \equiv 1/2, \quad t \in [-1, 1]. \quad (1.19)$$

The universality limit (1.17) holds for this  $V$  and thus takes the shape

$$\lim_{N \rightarrow \infty} \frac{2}{N} K_N \left( \frac{2x}{N}, \frac{2y}{N}; e^{-NV} \right) = \frac{\sin \pi(x-y)}{\pi(x-y)}. \quad (1.20)$$

The limit (1.16) for  $w_R$  can be understood from comparison with (1.20), since it turns out that  $w_R \approx e^{-NV}$  in the sense that as  $N, R \rightarrow \infty$  with  $N/R \rightarrow 2$

$$\lim_{R \rightarrow \infty} -\frac{1}{N} \log w_R(t) = NV(t), \quad t \in (-1, 1). \quad (1.21)$$

The limit (1.21) follows from the inequalities given in Proposition 1.5 below.

In what follows we assume that  $R, N$  are given in such a way that  $N/R \rightarrow 2$  as  $R \rightarrow \infty$ , and we may write  $\lim_{R \rightarrow \infty}$  and  $\lim_{N \rightarrow \infty}$  interchangeably. For every  $R > 0$  we define

$$\varepsilon_R = \frac{2R}{N} \sum_{|p_n| > R} \frac{1}{p_n}. \quad (1.22)$$

Because of the assumption (1.6) and (1.15) we have  $\varepsilon_R \rightarrow 0$  as  $R \rightarrow \infty$ .

The following proposition will be proven in section 2.

**Proposition 1.5** (a) *For every  $\alpha > 1$ , there is  $R_\alpha > 0$  such that if  $R \geq R_\alpha$ , then*

$$w_R(t) \leq e^{-N(V(\frac{t}{\alpha}) + \varepsilon_R t)}, \quad t \in [-1, 1]. \quad (1.23)$$

(b) *For every  $\alpha > 1$  and  $\beta \in (0, 1)$ , there is  $R_{\alpha, \beta} > 0$  such that if  $R \geq R_{\alpha, \beta}$ , then*

$$w_R(t) \geq e^{-N(V(\alpha t) + \varepsilon_R t)}, \quad t \in [-\beta, \beta]. \quad (1.24)$$

Note that equality holds in (1.23) and (1.24) at  $t = 0$ . The number (1.22) is such that the derivatives at  $t = 0$  are the same as well, which is of course necessary in order to have the inequalities (1.23) and (1.24) with equality at  $t = 0$ , since  $V(0) = V'(0) = 0$ .

Based on (1.23) and (1.24) we introduce for every  $R > 0$  and every  $\alpha > 1$  the following two weights on  $[-1, 1]$ :

$$w_{R, \alpha}^+(t) = (1 - t^2)^{-1/2} e^{-N(V(\frac{t}{\alpha}) + \varepsilon_R t)}, \quad (1.25)$$

$$w_{R, \alpha}^-(t) = (1 - \beta^{-2} t^2)^{1/2} e^{-N(V(\alpha t) + \varepsilon_R t)} \chi_{[-\beta, \beta]}(t), \quad \beta = \alpha^{-2}, \quad (1.26)$$

where we introduced the square root prefactors for convenience of further asymptotic analysis.

The weights  $w_{R, \alpha}^\pm$  are sufficiently nice, and with known techniques we can find the universality limits for them. This is stated in part (b) of the next proposition. Part (a) follows almost immediately from Proposition 1.5 and the definitions.

**Proposition 1.6** (a) *For each  $\alpha > 1$ , there is a number  $R_\alpha$  such that for  $R \geq R_\alpha$ ,*

$$w_{R, \alpha}^-(t) \leq w_R(t) \leq w_{R, \alpha}^+(t), \quad t \in [-1, 1], \quad (1.27)$$

and

$$\lim_{R \rightarrow \infty} w_{R, \alpha}^-\left(\frac{x}{R}\right) = \lim_{R \rightarrow \infty} w_{R, \alpha}^+\left(\frac{x}{R}\right) = 1 \quad (1.28)$$

uniformly for  $x$  in compact sets.

(b) There exist constants  $c_\alpha^\pm > 0$  such that for every  $\alpha > 1$ ,

$$\lim_{R \rightarrow \infty} \frac{1}{N} K_N \left( \frac{x}{N}, \frac{y}{N}; w_{R,\alpha}^\pm \right) = \frac{\sin \pi c_\alpha^\pm (x - y)}{\pi(x - y)} \quad (1.29)$$

uniformly for  $x$  and  $y$  in compact sets, and the constants  $c_\alpha^\pm$  satisfy

$$\lim_{\alpha \rightarrow 1+} c_\alpha^- = \lim_{\alpha \rightarrow 1+} c_\alpha^+ = \frac{1}{2}. \quad (1.30)$$

Proposition 1.6 is proved in section 3.

In section 4 we turn to the proof of our main Theorem 1.4. We use ideas of Lubinsky [20] in order to compare the kernel  $K_N(x, y; w_R)$  with those for the weights  $w_{R,\alpha}^\pm$ . The properties listed in Proposition 1.6 turn out to be enough to prove Theorem 1.4, and this in turn gives us Theorem 1.3 as we already mentioned.

## 2 Proof of Proposition 1.5

### 2.1 A lemma

We start the proof of Proposition 1.5 with a lemma.

**Lemma 2.1** *Suppose  $R$  is a positive integer. Then,*

$$\prod_{n=R}^{\infty} \left( 1 - \frac{R^2 t^2}{n^2} \right)^2 \leq e^{-2RV(t)} \leq \prod_{n=R+1}^{\infty} \left( 1 - \frac{R^2 t^2}{n^2} \right)^2, \quad t \in [-1, 1]. \quad (2.1)$$

**Proof.** Let  $t \in [-1, 1]$ . The function  $x \mapsto \psi(x) = -\log \left( 1 - \frac{R^2 t^2}{x^2} \right)$  is positive and decreasing for  $x \geq R$ . Therefore, by comparing the integral with Riemann sums, we have

$$\sum_{n=R+1}^{\infty} \psi(n) \leq \int_R^{\infty} \psi(x) dx \leq \sum_{n=R}^{\infty} \psi(n),$$

since  $R$  is an integer. Since  $\int_R^{\infty} \psi(x) dx = RV(t)$ , we get

$$-\sum_{n=R}^{\infty} \psi(n) \leq -RV(t) \leq -\sum_{n=R+1}^{\infty} \psi(n).$$

The inequality (2.1) follows by taking exponentials and squaring the expressions.  $\square$



## 2.2 Proof of Proposition 1.5 (a)

**Proof.** Take  $\alpha > 1$ . Since  $2R/N \rightarrow 1$  as  $R \rightarrow \infty$  and  $p_n/n \rightarrow 1$  as  $n \rightarrow \infty$ , there is  $R_\alpha$  such that for  $R \geq R_\alpha$  we have

$$\frac{2R}{\alpha} < N < 2\alpha R, \quad (2.2)$$

and if  $|n| \geq \lfloor \alpha R \rfloor$ , then

$$\frac{1}{\alpha} \leq \frac{p_n}{n} \leq \alpha, \quad (2.3)$$

and

$$|p_n| > R. \quad (2.4)$$

We then write for  $R \geq R_\alpha$  and  $t \in [-1, 1]$ ,

$$\begin{aligned} w_R(t) &= \prod_{|p_n| \geq R} \left(1 - \frac{Rt}{p_n}\right)^2 \\ &= \prod_{n \in S_1} \left(1 - \frac{Rt}{p_n}\right)^2 \left( \prod_{n \in S_2 \cup S_3} \left(1 - \frac{Rt}{p_n}\right)^2 \left(1 - \frac{Rt}{p_{-n}}\right)^2 \right) \end{aligned} \quad (2.5)$$

where the sets  $S_1$ ,  $S_2$  and  $S_3$  are defined as follows

$$\begin{aligned} S_1 &= \{n \in \mathbb{Z} \mid |p_n| > R \text{ and } |n| < \lfloor \alpha R \rfloor\}, \\ S_2 &= \{n \in \mathbb{N} \mid n \geq \lfloor \alpha R \rfloor \text{ and } p_n \geq -p_{-n}\}, \\ S_3 &= \{n \in \mathbb{N} \mid n \geq \lfloor \alpha R \rfloor \text{ and } p_n < -p_{-n}\}. \end{aligned} \quad (2.6)$$

Because of (2.4) and the above definitions (2.6), we have a disjoint union

$$\{n \mid |p_n| > R\} = S_1 \cup S_2 \cup (-S_2) \cup S_3 \cup (-S_3), \quad (2.7)$$

which is used in the factorization (2.5). Also  $S_1$  is a finite set.

For  $n \in S_1$ , we use the estimate  $1 - \frac{Rt}{p_n} \leq e^{-\frac{Rt}{p_n}}$  which follows from the elementary inequalities  $0 \leq 1 + x \leq e^x$  if  $x \geq -1$ . Thus for  $t \in [-1, 1]$ ,

$$\prod_{n \in S_1} \left(1 - \frac{Rt}{p_n}\right)^2 \leq e^{-2Rt \sum_{n \in S_1} \frac{1}{p_n}}. \quad (2.8)$$

For  $n \in S_2$ , we write for  $t \in [-1, 1]$ ,

$$\begin{aligned} \left(1 - \frac{Rt}{p_n}\right) \left(1 - \frac{Rt}{p_{-n}}\right) &= \left(1 - \frac{Rt}{p_n}\right) \left(1 + \frac{Rt}{p_n} - Rt \left(\frac{1}{p_n} + \frac{1}{p_{-n}}\right)\right) \\ &= \left(1 - \frac{R^2 t^2}{p_n^2}\right) \left(1 + \frac{Rt}{1 + \frac{Rt}{p_n}} \left(-\frac{1}{p_n} - \frac{1}{p_{-n}}\right)\right). \end{aligned} \quad (2.9)$$

Since  $p_n \geq -p_{-n} \geq R$  by (2.4) and the definition of  $S_2$  in (2.6), we have  $-\frac{1}{p_n} - \frac{1}{p_{-n}} \geq 0$ , and we can use  $\frac{Rt}{1+\frac{Rt}{p_n}} \leq Rt$  for  $t \in [-1, 1]$ , to obtain from (2.9)

$$\begin{aligned} \left(1 - \frac{Rt}{p_n}\right) \left(1 - \frac{Rt}{p_{-n}}\right) &\leq \left(1 - \frac{R^2 t^2}{p_n^2}\right) \left(1 + Rt \left(-\frac{1}{p_n} - \frac{1}{p_{-n}}\right)\right) \\ &\leq \left(1 - \frac{R^2 t^2}{p_n^2}\right) e^{-Rt \left(\frac{1}{p_n} + \frac{1}{p_{-n}}\right)}, \end{aligned} \quad (2.10)$$

where we used again that  $1 + x \leq e^x$ . Since  $p_n \leq n\alpha$  by (2.3), we can further estimate this by

$$\left(1 - \frac{Rt}{p_n}\right) \left(1 - \frac{Rt}{p_{-n}}\right) \leq \left(1 - \frac{R^2 t^2}{\alpha^2 n^2}\right) e^{-Rt \left(\frac{1}{p_n} + \frac{1}{p_{-n}}\right)}. \quad (2.11)$$

The estimates for  $n \in S_3$  are similar. We interchange the roles of  $p_n$  and  $p_{-n}$  in (2.9), we use  $\frac{Rt}{1+\frac{Rt}{p_{-n}}} \geq Rt$  for  $t \in [-1, 1]$  to obtain (2.10) with  $p_n$  interchanged with  $p_{-n}$ . Since  $0 > p_{-n} \geq -n\alpha$ , we also find (2.11) for  $n \in S_3$ . Hence

$$\prod_{n \in S_2 \cup S_3} \left(1 - \frac{Rt}{p_n}\right)^2 \left(1 - \frac{Rt}{p_{-n}}\right)^2 \leq \prod_{n \in S_2 \cup S_3} \left(1 - \frac{R^2 t^2}{\alpha^2 n^2}\right)^2 e^{-2Rt \sum_{n \in S_2 \cup S_3} \left(\frac{1}{p_n} + \frac{1}{p_{-n}}\right)}. \quad (2.12)$$

Using (2.8) and (2.12) in (2.5), we obtain

$$\begin{aligned} w_R(t) &\leq \left( \prod_{n \in S_2 \cup S_3} \left(1 - \frac{R^2 t^2}{\alpha^2 n^2}\right)^2 \right) e^{-2Rt \left( \sum_{n \in S_1} \frac{1}{p_n} + \sum_{n \in S_2 \cup S_3} \left(\frac{1}{p_n} + \frac{1}{p_{-n}}\right) \right)} \\ &= \left( \prod_{n=\lfloor \alpha R \rfloor}^{\infty} \left(1 - \frac{R^2 t^2}{\alpha^2 n^2}\right)^2 \right) e^{-N\varepsilon_R t}, \end{aligned} \quad (2.13)$$

where in the last step we used (1.22) and the property (2.7).

Then we use the first inequality in Lemma 2.1 with  $\lfloor \alpha R \rfloor$  instead of  $R$  and  $t/\alpha$  instead of  $t$ . Then by (2.1) and (2.13),

$$w_R(t) \leq e^{-2\lfloor \alpha R \rfloor V\left(\frac{t}{\alpha}\right)} e^{-N\varepsilon_R t}.$$

We finally use (2.2) and we obtain the required inequality (1.23).  $\square$

### 2.3 Proof of Proposition 1.5 (b)

**Proof.** The proof for the lower estimate (1.24) follows along the same lines, but it is more involved. Instead of the inequality  $1 + x \leq e^x$ , we now use a less obvious, but still elementary, inequality

$$1 + x \geq e^{x - \frac{x^2}{1+x}}, \quad x > -1. \quad (2.14)$$

It follows from (2.14) that for any  $\beta \in (0, 1)$ ,

$$1 + x \geq e^{x - \frac{1}{1-\beta}x^2}, \quad x \geq -\beta. \quad (2.15)$$

Then, fix  $\alpha > 1$ ,  $\beta \in (0, 1)$ , and take  $\gamma \in (1, 2)$  so close to 1 that

$$\gamma^2 + 12 \frac{\gamma^2 - 1}{1 - \beta} < \alpha^2. \quad (2.16)$$

As in the proof of the upper estimate we assume that  $R$  is large enough so that (2.2), (2.3), (2.4) hold, but with  $\alpha$  replaced by  $\gamma$ . Thus we take  $R_\gamma > \frac{4}{\gamma^2 - 1}$  big enough such that for  $R \geq R_\gamma$  we have

$$\frac{2R}{\gamma} < N < 2\gamma R, \quad (2.17)$$

and if  $|n| \geq \lfloor \gamma R \rfloor$ , then

$$\frac{1}{\gamma} \leq \frac{p_n}{n} \leq \gamma, \quad (2.18)$$

and

$$|p_n| > R. \quad (2.19)$$

Since  $R_\gamma > \frac{4}{\gamma^2 - 1}$ , we also have that for  $R \geq R_\gamma$

$$R \geq \frac{4}{\gamma^2 - 1}. \quad (2.20)$$

We fix  $R \geq R_\gamma$  and  $t \in [-\beta, \beta]$ . We use the factorization of  $w_R$  as in (2.5), but with slightly different sets  $S_1, S_2, S_3$ , namely

$$\begin{aligned} S_1^* &= \{n \in \mathbb{Z} \mid |p_n| > R \text{ and } |n| \leq \lceil \gamma R \rceil\}, \\ S_2^* &= \{n \in \mathbb{N} \mid n \geq \lceil \gamma R \rceil + 1 \text{ and } p_n \geq -p_{-n}\}, \\ S_3^* &= \{n \in \mathbb{N} \mid n \geq \lceil \gamma R \rceil + 1 \text{ and } p_n < -p_{-n}\}. \end{aligned} \quad (2.21)$$

For  $n \in S_1^*$  we obtain from (2.15) that for  $t \in [-\beta, \beta]$

$$1 - \frac{Rt}{p_n} \geq e^{-\frac{Rt}{p_n} - \frac{(Rt)^2}{(1-\beta)p_n^2}},$$

since  $|\frac{Rt}{p_n}| \leq |t| \leq \beta$ , and therefore

$$\prod_{n \in S_1^*} \left(1 - \frac{Rt}{p_n}\right)^2 \geq e^{-2Rt \sum_{n \in S_1} \frac{1}{p_n} - 2\frac{R^2 t^2}{1-\beta} \sum_{n \in S_1} \frac{1}{p_n^2}}. \quad (2.22)$$

For  $n \in S_2^*$ , we do not use (2.9) but rather

$$\left(1 - \frac{Rt}{p_n}\right) \left(1 - \frac{Rt}{p_{-n}}\right) = \left(1 - \frac{R^2 t^2}{p_{-n}^2}\right) \left(1 + \frac{Rt}{1 + \frac{Rt}{p_{-n}}} \left(-\frac{1}{p_n} - \frac{1}{p_{-n}}\right)\right). \quad (2.23)$$

Since  $p_{-n} \leq -R$ , see (2.19), we have  $\frac{Rt}{1 + \frac{Rt}{p_{-n}}} \geq Rt$ , and we obtain since  $-\frac{1}{p_n} - \frac{1}{p_{-n}} \geq 0$ ,

$$\left(1 - \frac{Rt}{p_n}\right) \left(1 - \frac{Rt}{p_{-n}}\right) \geq \left(1 - \frac{R^2 t^2}{p_{-n}^2}\right) \left(1 + Rt \left(-\frac{1}{p_n} - \frac{1}{p_{-n}}\right)\right). \quad (2.24)$$

Since  $p_n \geq -p_{-n} \geq R$  we then have

$$\left|Rt \left(-\frac{1}{p_n} - \frac{1}{p_{-n}}\right)\right| \leq \left|\frac{Rt}{p_{-n}}\right| \leq |t| \leq \beta,$$

and therefore we can apply (2.15) to estimate the last factor in (2.24). In the first factor we use  $p_{-n}^2 \geq \left(\frac{n}{\gamma}\right)^2$ , see (2.18), and we obtain from (2.24)

$$\left(1 - \frac{Rt}{p_n}\right) \left(1 - \frac{Rt}{p_{-n}}\right) \geq \left(1 - \frac{\gamma^2 R^2 t^2}{n^2}\right) e^{-Rt \left(\frac{1}{p_n} + \frac{1}{p_{-n}}\right) - \frac{R^2 t^2}{1-\beta} R^2 t^2 \left(\frac{1}{p_n} + \frac{1}{p_{-n}}\right)^2}. \quad (2.25)$$

The same estimate (2.25) holds for  $n \in S_3^*$  as well (we find it by interchanging the roles of  $p_n$  and  $p_{-n}$  in (2.24)), and we obtain

$$\begin{aligned} \prod_{n \in S_2^* \cup S_3^*} \left(1 - \frac{Rt}{p_n}\right) \left(1 - \frac{Rt}{p_{-n}}\right)^2 &\geq \prod_{n \in S_2^* \cup S_3^*} \left(1 - \frac{\gamma^2 R^2 t^2}{n^2}\right)^2 \\ &\times e^{-2Rt \sum_{n \in S_2^* \cup S_3^*} \left(\frac{1}{p_n} + \frac{1}{p_{-n}}\right) - 2\frac{R^2 t^2}{1-\beta} \sum_{n \in S_2^* \cup S_3^*} \left(\frac{1}{p_n} + \frac{1}{p_{-n}}\right)^2}. \end{aligned} \quad (2.26)$$

Combining (2.5) (with  $S_j$  replaced by  $S_j^*$  for  $j = 1, 2, 3$ ), (2.22) and (2.26) we have for  $t \in [-\beta, \beta]$ ,

$$\begin{aligned}
w_R(t) &\geq \left( \prod_{n \in S_2^* \cup S_3^*} \left( 1 - \frac{\gamma^2 R^2 t^2}{n^2} \right)^2 \right) e^{-2Rt \left( \sum_{n \in S_1^*} \frac{1}{p_n} + \sum_{n \in S_2^* \cup S_3^*} \left( \frac{1}{p_n} + \frac{1}{p-n} \right) \right)} \\
&\quad \times e^{-\frac{2R^2}{1-\beta} \left( \sum_{n \in S_1^*} \frac{1}{p_n^2} + \sum_{n \in S_2^* \cup S_3^*} \left( \frac{1}{p_n} + \frac{1}{p-n} \right)^2 \right) t^2} \\
&= \left( \prod_{n=\lceil \gamma R \rceil+1}^{\infty} \left( 1 - \frac{\gamma^2 R^2 t^2}{n^2} \right)^2 \right) e^{-N \varepsilon_R t} \\
&\quad \times e^{-\frac{2R^2}{1-\beta} \left( \sum_{n \in S_1^*} \frac{1}{p_n^2} + \sum_{n \in S_2^* \cup S_3^*} \left( \frac{1}{p_n} + \frac{1}{p-n} \right)^2 \right) t^2}. \tag{2.27}
\end{aligned}$$

where we used (1.22).

It remains to estimate the first and third factor in the product in the right-hand side of (2.27).

To estimate the first factor we use the second inequality of (2.1) with  $\lceil \gamma R \rceil$  instead of  $R$  and  $\frac{\gamma R}{\lceil \gamma R \rceil} t$  instead of  $t$ . This gives us

$$\prod_{n=\lceil \gamma R \rceil+1}^{\infty} \left( 1 - \frac{\gamma^2 R^2 t^2}{n^2} \right)^2 \geq e^{-2\lceil \gamma R \rceil V(\frac{\gamma R}{\lceil \gamma R \rceil} t)} \tag{2.28}$$

Since  $V$  is convex and  $V(0) = 0$  (as can easily be checked from the definition (1.18)), we have

$$V\left(\frac{\gamma R t}{\lceil \gamma R \rceil}\right) \leq \frac{\gamma R}{\lceil \gamma R \rceil} V(t).$$

Using this in (2.28), and also recalling  $2R \leq \gamma N$ , see (2.17), we obtain

$$\prod_{n=\lceil \gamma R \rceil+1}^{\infty} \left( 1 - \frac{\gamma^2 R^2 t^2}{n^2} \right)^2 \geq e^{-\gamma^2 N V(t)}. \tag{2.29}$$

Next we estimate the third factor in (2.27). If  $n \in S_1^*$  then  $|p_n| > R$  and

$$\sum_{n \in S_1^*} \frac{1}{p_n^2} \leq \frac{\#S_1^*}{R^2}. \tag{2.30}$$

Also if  $n \in S_1^*$  we have  $|n| \leq \lceil \gamma R \rceil$  by the definition of  $S_1^*$  in (2.21), and  $|n| \geq |p_n|/\gamma \geq R/\gamma$ , where we also used (2.18). Thus  $S_1^*$  is contained in  $[-\gamma R - 1, -\frac{R}{\gamma}] \cup [\frac{R}{\gamma}, \gamma R + 1]$ . Since  $S_1^*$  contains integers only, we obtain

$$\#S_1^* \leq 2(\gamma R + 1 - \frac{R}{\gamma} + 1) < 2(\gamma^2 - 1)R + 4.$$

Combining this with (2.30) and (2.20) we find

$$\sum_{n \in S_1^*} \frac{1}{p_n^2} \leq \frac{2(\gamma^2 - 1)}{R} + \frac{4}{R^2} \leq \frac{3(\gamma^2 - 1)}{R}. \quad (2.31)$$

If  $n \in S_2^* \cup S_3^*$  then  $n \geq \lceil \gamma R \rceil + 1$  by (2.21), and both  $n/p_n$  and  $-n/p_{-n}$  lie between  $1/\gamma$  and  $\gamma$  by (2.18). Then

$$\left| \frac{1}{p_n} + \frac{1}{p_{-n}} \right| \leq \frac{\gamma - 1/\gamma}{n} < \frac{\gamma^2 - 1}{n}$$

and therefore

$$\begin{aligned} \sum_{n \in S_2^* \cup S_3^*} \left( \frac{1}{p_n} + \frac{1}{p_{-n}} \right)^2 &\leq \sum_{n=\lceil \gamma R \rceil+1}^{\infty} \left( \frac{\gamma^2 - 1}{n} \right)^2 \\ &\leq (\gamma^2 - 1)^2 \int_{\gamma R}^{\infty} \frac{dx}{x^2} \\ &\leq \frac{(\gamma^2 - 1)^2}{R} \\ &\leq \frac{3(\gamma^2 - 1)}{R}. \end{aligned} \quad (2.32)$$

In the last step we used  $\gamma^2 - 1 < 3$ . Using (2.31) and (2.32), we find the inequality

$$\sum_{n \in S_1^*} \frac{1}{p_n^2} + \sum_{n \in S_2^* \cup S_3^*} \left( \frac{1}{p_n} + \frac{1}{p_{-n}} \right)^2 \leq \frac{6(\gamma^2 - 1)}{R}.$$

Since  $2R \leq \gamma N$  with  $\gamma < 2$ , we have

$$2R^2 \frac{6(\gamma^2 - 1)}{R} \leq 12N(\gamma^2 - 1),$$

and therefore

$$\frac{2R^2}{1 - \beta} \left( \sum_{n \in S_1^*} \frac{1}{p_n^2} + \sum_{n \in S_2^* \cup S_3^*} \left( \frac{1}{p_n} + \frac{1}{p_{-n}} \right)^2 \right) \leq 12N \frac{\gamma^2 - 1}{1 - \beta}. \quad (2.33)$$

Having (2.29) and (2.33) we can continue with our lower bound for  $w_R(t)$ . From (2.27) we obtain

$$w_R(t) \geq e^{-\gamma^2 NV(t) - 12N \frac{\gamma^2 - 1}{1 - \beta} t^2} e^{-N\varepsilon_R t}. \quad (2.34)$$

From the definition of  $V$  in (1.18), we find the Maclaurin series

$$V(t) = t^2 + \sum_{k=2}^{\infty} \frac{t^{2k}}{k(2k-1)} \quad (2.35)$$

from which it is clear that  $t^2 \leq V(t)$ . Thus by (2.16),

$$\gamma^2 NV(t) + 12N \frac{\gamma^2 - 1}{1 - \beta} t^2 \leq \left( \gamma^2 + 12 \frac{\gamma^2 - 1}{1 - \beta} \right) NV(t) \leq \alpha^2 NV(t).$$

From the Maclaurin series (2.35) it is also clear that  $\alpha^2 V(t) \leq V(\alpha t)$ , since  $\alpha > 1$ . Hence

$$\gamma^2 NV(t) + 12N \frac{\gamma^2 - 1}{1 - \beta} t^2 \leq NV(\alpha t),$$

and using this in (2.34), we obtain the required inequality (1.24).  $\square$

### 3 Proof of Proposition 1.6

#### 3.1 Proof of Proposition 1.6 (a)

**Proof.** The inequalities in (1.27) follow from Proposition 1.5 and the definitions (1.25)-(1.26).

The limits in (1.28) are also almost immediate. From (1.25) we have

$$w_{R,\alpha}^+ \left( \frac{x}{R} \right) = \left( 1 - \frac{x^2}{R^2} \right)^{-1/2} \exp \left( -NV \left( \frac{x}{\alpha R} \right) - \varepsilon_R \frac{Nx}{R} \right). \quad (3.1)$$

Clearly  $\left( 1 - \frac{x^2}{R^2} \right)^{-1/2} \rightarrow 1$  as  $R \rightarrow \infty$ , and by (1.22)

$$\varepsilon_R \frac{Nx}{R} = 2x \sum_{|p_n| > R} \frac{1}{p_n},$$

which tends to 0 as  $R \rightarrow \infty$ .

From (2.35) we see that  $V(t) = O(t^2)$  as  $t \rightarrow 0$  and therefore

$$NV \left( \frac{x}{\alpha R} \right) = NO \left( \left( \frac{x}{\alpha R} \right)^2 \right) = \frac{x^2}{\alpha^2} O \left( \frac{N}{R^2} \right)$$

and this also tends to 0 as  $R \rightarrow \infty$ . Thus (3.1) tends to 1 as  $R \rightarrow \infty$ , and the convergence is uniform for  $x$  in a compact set. This proves the second limit in (1.28). The proof for the first limit is the same.  $\square$

### 3.2 Preliminaries to the proof of Proposition 1.6 (b)

To establish the limits (1.29) for the weights  $w_{R,\alpha}^\pm$  we need a few concepts of logarithmic potential theory with external fields [24].

Let  $W$  be a continuous function on a compact interval  $I$  of the real line. Among all probability measures  $\mu$  with  $\text{supp}(\mu) \subset I$ , there exists a unique measure  $\mu_W$ , called the equilibrium measure in the presence of the external field  $W$ , which minimizes the weighted energy

$$\iint \log \frac{1}{|x-t|} d\mu(x) d\mu(t) + \int W(t) d\mu(t).$$

If we denote by  $U^\mu$  the logarithmic potential of a measure  $\mu$ , that is,

$$U^\mu(x) = \int \log \frac{1}{|x-t|} d\mu(t), \quad (3.2)$$

then  $\mu_W$  is characterized by the property that

$$\begin{aligned} 2U^{\mu_W}(x) + W(x) &\geq \ell, & x \in I \setminus \text{supp}(\mu_W), \\ 2U^{\mu_W}(x) + W(x) &= \ell, & x \in \text{supp}(\mu_W), \end{aligned} \quad (3.3)$$

for some constant  $\ell$  that depends on  $W$ .

We will specifically consider external fields of the form

$$V_{\alpha,\varepsilon}(x) = V\left(\frac{x}{\alpha}\right) + \varepsilon x, \quad (3.4)$$

where  $\alpha \geq 1$  and  $\varepsilon$  are real numbers, and  $V$  is given by (1.18) as before. We use  $\psi_{\alpha,\varepsilon}$  to denote the density of the equilibrium measure with external field (3.4) on the interval  $[-1, 1]$ . We already noted in (1.19) that

$$\psi_{1,0}(x) = \frac{1}{2}, \quad x \in [-1, 1]. \quad (3.5)$$

For  $\alpha > 1$  and  $\varepsilon$  close to zero we can also calculate the density explicitly. We write

$$\varepsilon_\alpha := 2\sqrt{1 - \alpha^{-2}}. \quad (3.6)$$

**Proposition 3.1** *Let  $\alpha > 1$  and  $\varepsilon \in [-\varepsilon_\alpha, \varepsilon_\alpha]$ . Then the equilibrium measure  $\mu_{V_{\alpha,\varepsilon}}$  in the presence of the external field (3.4) has full support  $[-1, 1]$  with density given by*

$$\psi_{\alpha,\varepsilon}(x) = \frac{2\sqrt{\alpha^2 - 1} - \alpha\varepsilon x}{2\alpha\pi\sqrt{1 - x^2}} + \frac{1}{\alpha\pi} \arctan\left(\frac{\sqrt{1 - x^2}}{\sqrt{\alpha^2 - 1}}\right), \quad x \in (-1, 1). \quad (3.7)$$



**Proof.** Note that the density (3.7) is indeed positive on  $(-1, 1)$  because of the assumption  $|\varepsilon| \leq \varepsilon_\alpha$ , see (3.6). The density is zero at one of the endpoints in case  $|\varepsilon| = \varepsilon_\alpha$ .

We first consider  $\varepsilon = 0$ .

The equilibrium measure for the external field  $V(x/\alpha)$  on the bigger interval  $[-\alpha, \alpha]$  is a multiple of the Lebesgue measure, namely  $\frac{1}{2\alpha}dx$  restricted to  $[-\alpha, \alpha]$ . This simply follows from (3.5) by rescaling. The equilibrium measure  $\mu_{V_{\alpha,0}}$  is then the balayage of  $\frac{1}{2\alpha}dx$  onto  $[-1, 1]$ . The balayage of a point mass  $\delta_t$  onto  $[-1, 1]$  is the measure with density

$$\frac{1}{\pi\sqrt{1-x^2}} \frac{\sqrt{t^2-1}}{|t-x|} dx, \quad \text{for } t > 1 \text{ or } t < -1.$$

It follows that

$$\begin{aligned} \psi_{\alpha,0}(x) &= \frac{1}{2\alpha} + \frac{1}{2\alpha\pi\sqrt{1-x^2}} \left( \int_{-\alpha}^{-1} + \int_1^{\alpha} \right) \frac{\sqrt{t^2-1}}{|t-x|} dt, \\ &= \frac{1}{2\alpha} + \frac{1}{\alpha\pi\sqrt{1-x^2}} \int_1^{\alpha} \frac{t\sqrt{t^2-1}}{t^2-x^2} dt, \quad x \in [-1, 1]. \end{aligned}$$

A change of variables  $t^2 = s^2 + 1$  leads to

$$\psi_{\alpha,0}(x) = \frac{1}{2\alpha} + \frac{1}{\alpha\pi\sqrt{1-x^2}} \int_0^{\sqrt{\alpha^2-1}} \frac{s^2}{s^2+1-x^2} ds, \quad x \in [-1, 1].$$

The integral can be evaluated, and the result is

$$\psi_{\alpha,0}(x) = \frac{1}{2\alpha} + \frac{1}{\alpha\pi\sqrt{1-x^2}} \left[ \sqrt{\alpha^2-1} - \sqrt{1-x^2} \arctan \left( \frac{\sqrt{\alpha^2-1}}{\sqrt{1-x^2}} \right) \right], \quad (3.8)$$

for  $x \in [-1, 1]$ . Using  $\arctan(y) + \arctan(1/y) = \pi/2$ , we see that (3.8) leads to (3.7) for the case  $\varepsilon = 0$ .

Let now  $\nu$  be the signed measure on  $[-1, 1]$  given by

$$d\nu = \frac{x}{\pi\sqrt{1-x^2}} dx. \quad (3.9)$$

We claim that

$$U^\nu(x) = x, \quad x \in [-1, 1]. \quad (3.10)$$

To prove (3.10) we recall that the measure  $\omega$  with density

$$\frac{d\omega}{dx} = \frac{1}{\pi\sqrt{1-x^2}}, \quad x \in [-1, 1]$$

is the equilibrium measure of the interval  $[-1, 1]$ . Thus  $U^\omega$  is constant on  $[-1, 1]$  and therefore  $\frac{d}{dx}U^\omega(x) = 0$  for  $x \in (-1, 1)$ . This means

$$-\frac{1}{\pi} \int_{-1}^1 \frac{1}{x-t} \frac{1}{\sqrt{1-t^2}} dt = 0, \quad x \in [-1, 1],$$

where  $\int$  denotes the Cauchy principal value. Hence

$$\begin{aligned} \frac{d}{dx}U^\nu(x) &= -\frac{1}{\pi} \int_{-1}^1 \frac{1}{x-t} \frac{t}{\sqrt{1-t^2}} dt \\ &= \frac{1}{\pi} \int_{-1}^1 \frac{1}{x-t} \frac{x-t-x}{\sqrt{1-t^2}} dt \\ &= \int_{-1}^1 d\omega(t) + x \frac{d}{dx}U^\omega(x) \\ &= 1. \end{aligned}$$

Thus  $U^\nu(x) = x + c$ ,  $x \in [-1, 1]$ , for some constant  $c$ . By symmetry, it is clear that  $U^\nu(0) = 0$ , and therefore  $c = 0$ . Hence our claim (3.10) follows.

Then  $\mu_{\alpha,0} - \frac{\varepsilon}{2}\nu$  is a measure with density (3.7). It is a positive measure on  $[-1, 1]$  if  $|\varepsilon| \leq \varepsilon_\alpha$ , and then it is a probability measure since  $\int d\nu = 0$  and  $\int d\mu_{\alpha,0} = 1$ . It satisfies for some constant  $\ell$ ,

$$\begin{aligned} 2U^{\mu_{\alpha,0} - \frac{\varepsilon}{2}\nu}(x) &= 2U^{\mu_{\alpha,0}}(x) - \varepsilon U^\nu(x) \\ &= -V\left(\frac{x}{\alpha}\right) + \ell - \varepsilon x, \quad x \in [-1, 1]. \end{aligned}$$

Because of the variational conditions (3.3) this means that

$$\mu_{\alpha,\varepsilon} = \mu_{\alpha,0} - \frac{\varepsilon}{2}\nu$$

and the density (3.7) follows.  $\square$

Note that for  $|\varepsilon| \leq \varepsilon_\alpha$ ,

$$\psi_{\alpha,\varepsilon}(0) = \psi_{\alpha,0}(0) = \frac{1}{2\alpha} + \frac{\sqrt{\alpha^2 - 1} - \arctan(\sqrt{\alpha^2 - 1})}{\alpha\pi}, \quad (3.11)$$

which can be seen from (3.7) and (3.8). Also note that  $\psi_{\alpha,\varepsilon}(0) \rightarrow \frac{1}{2}$  as  $\alpha \rightarrow 1+$ .

Proposition 1.6 (b) will follow from the following universality result for the weights  $(1-x^2)^{\pm 1/2} e^{-NV_{\alpha,\varepsilon}(x)}$ .

**Proposition 3.2** *Let  $\alpha > 1$  and  $|\varepsilon| < \varepsilon_\alpha$ , where  $\varepsilon_\alpha$  is given by (3.6). Let  $\psi_{\alpha,\varepsilon}$  be the density of the equilibrium measure in  $[-1, 1]$  for the external field (3.4) on  $[-1, 1]$ .*

*Then for each  $x_0 \in (-1, 1)$ ,*

$$\begin{aligned} \frac{1}{\psi_{\alpha,\varepsilon}(x_0)N} K_N \left( x_0 + \frac{x}{\psi_{\alpha,\varepsilon}(x_0)N}, x_0 + \frac{y}{\psi_{\alpha,\varepsilon}(x_0)N}; (1-t^2)^{-1/2} e^{-NV_{\alpha,\varepsilon}(t)} \right) \\ = \frac{\sin \pi(x-y)}{\pi(x-y)} + O\left(\frac{1}{N}\right) \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} \frac{1}{\psi_{\alpha,\varepsilon}(x_0)N} K_N \left( x_0 + \frac{x}{\psi_{\alpha,\varepsilon}(x_0)N}, x_0 + \frac{y}{\psi_{\alpha,\varepsilon}(x_0)N}; (1-t^2)^{1/2} e^{-NV_{\alpha,\varepsilon}(t)} \right) \\ = \frac{\sin \pi(x-y)}{\pi(x-y)} + O\left(\frac{1}{N}\right) \end{aligned} \quad (3.13)$$

as  $N \rightarrow \infty$ .

The  $O$ -terms in (3.12) and (3.13) are uniform for  $x$  and  $y$  in a compact subset of the real line, for  $x_0$  in a compact subset of  $(-1, 1)$ , and for  $\varepsilon$  in a compact subset of  $(-\varepsilon_\alpha, \varepsilon_\alpha)$ .

The universality results (3.12) and (3.13) are well-known and they are in fact known under much more general conditions. The interest for our present purposes lies in the fact that the  $O$ -terms are uniform in  $\varepsilon$ .

Deift et al. [10] proved universality results for weights  $e^{-nW}$  on the real line and they developed the Riemann-Hilbert method for orthogonal polynomials [9] in order to do so. For nonvarying weights on  $[-1, 1]$  that are real analytic modifications of a Jacobi weight, i.e., weights of the form

$$h(x)(1-x)^\alpha(1+x)^\beta \quad (3.14)$$

where  $h$  is real analytic and nowhere vanishing on  $[-1, 1]$ , the universality was shown in [18]. In [17] it was emphasized that the Riemann-Hilbert method does not require any endpoint analysis for weights of the form (3.14) with  $\alpha, \beta \in \{-\frac{1}{2}, \frac{1}{2}\}$ .

Varying weights  $h(x)(1-x)^\alpha(1+x)^\beta e^{-nW(x)}$  on the interval  $[-1, 1]$  have not been considered explicitly, as far as we are aware, but the Riemann-Hilbert analysis is very similar to the one in [18], provided that  $W$  is real analytic in a neighborhood of  $[-1, 1]$  with an equilibrium measure having a density  $\psi_W$  that is nowhere vanishing on  $(-1, 1)$  and has inverse square root behavior at both endpoints  $\pm 1$ . Again, if  $\alpha, \beta \in \{-\frac{1}{2}, \frac{1}{2}\}$ , then no endpoint

analysis in the Riemann-Hilbert method is needed. This is a technical advantage, and it is the reason for the prefactors  $(1 - x^2)^{\pm 1/2}$  in the weights in (3.12) and (3.13).

In another development, Levin and Lubinsky [19, Theorem 1.1] established the sine kernel universality limits for very general exponential weights  $he^{-nW}$ , with fixed  $h$  and  $W$  that do not depend on  $n$ . We cannot apply this result directly, since we need uniformity in  $\varepsilon$ . In the same paper [19, Theorem 1.2], universality for exponential weights  $he^{-nW_n}$  with varying external fields  $W_n$  is established under the assumption that some control on the behavior of the Christoffel functions (i.e., the kernel on the diagonal) is known a priori. The behavior of Christoffel functions has been obtained by Totik [25, Theorems 1.1 and 1.2], but again uniformity in parameters was not included.

Rather than trying to adapt the arguments of Totik to our situation, which deals with a very specific weight, we rely on the Riemann-Hilbert method to prove Proposition 3.2. The application of this method is standard by now, but we will go through the analysis in order to verify the uniformity in  $\varepsilon$ , see appendix A.

### 3.3 Proof of Proposition 1.6 (b)

We finally show how part (b) of Proposition 1.6 follows from Proposition 3.2.

**Proof.** Specializing (3.12) to  $x_0 = 0$ , putting

$$c_\alpha^+ = \psi_{\alpha,\varepsilon}(0) = \psi_{\alpha,0}(0), \quad (3.15)$$

see (3.11), and changing  $x$  and  $y$  to  $c_\alpha^+x$  and  $c_\alpha^+y$ , we obtain

$$\lim_{N \rightarrow \infty} \frac{1}{N} K_N \left( \frac{x}{N}, \frac{y}{N}; (1 - t^2)^{-1/2} e^{-NV_{\alpha,\varepsilon}(t)} \right) = \frac{\sin \pi c_\alpha^+(x - y)}{\pi(x - y)}$$

uniformly for  $\varepsilon$  in compact subsets of  $(-\varepsilon_\alpha, \varepsilon_\alpha)$ . By the uniformity, we obtain the same limit, if we let  $\varepsilon = \varepsilon_R$  and let both  $N, R \rightarrow \infty$  such that  $\varepsilon_R \rightarrow 0$ . Hence (1.29) holds for the  $+$ -case.

The weight  $w_{R,\alpha}^-$  is supported on  $[-\alpha^{-2}, \alpha^{-2}]$ , see (1.26). After rescaling to  $[-1, 1]$  it is

$$w_{R,\alpha}^-(\alpha^{-2}t) = (1 - t^2)^{1/2} e^{-N(V(\frac{t}{\alpha}) + \tilde{\varepsilon}_R t)}, \quad t \in [-1, 1],$$

where

$$\tilde{\varepsilon}_R = \alpha^{-2} \varepsilon_R.$$

The kernel transforms as

$$K_N(x, y, w_{R,\alpha}^-) = \alpha^2 K_N(\alpha^2 x, \alpha^2 y; (1-t^2)^{1/2} e^{-NV_{\alpha,\varepsilon_R}(t)}). \quad (3.16)$$

From (3.13) with  $x_0 = 0$ ,  $x$  and  $y$  replaced by  $\psi_{\alpha,\varepsilon}(0)x\alpha^2$  and  $\psi_{\alpha,\varepsilon}(0)y\alpha^2$ , we obtain

$$\lim_{N \rightarrow \infty} \frac{\alpha^2}{N} K_N \left( \frac{x\alpha^2}{N}, \frac{y\alpha^2}{N}; (1-t^2)^{1/2} e^{-NV_{\alpha,\varepsilon}(t)} \right) = \frac{\sin \pi c_\alpha^-(x-y)}{\pi(x-y)}$$

uniformly for  $\varepsilon$  in compact subsets of  $(-\varepsilon_\alpha, \varepsilon_\alpha)$ , where

$$c_\alpha^- = \alpha^2 \psi_{\alpha,0}(0). \quad (3.17)$$

Because of the uniformity in  $\varepsilon$ , we get

$$\lim_{R \rightarrow \infty} \frac{\alpha^2}{N} K_N \left( \frac{x\alpha^2}{N}, \frac{y\alpha^2}{N}; (1-t^2)^{1/2} e^{-NV_{\alpha,\tilde{\varepsilon}_R}(t)} \right) = \frac{\sin \pi c_\alpha^-(x-y)}{\pi(x-y)} \quad (3.18)$$

if both  $R, N \rightarrow \infty$  and  $\tilde{\varepsilon}_R \rightarrow 0$ . Combining (3.16) and (3.18) we obtain (1.29) for the  $--$  case as well.

The limits (1.30) are immediate from (3.15) and (3.17), since  $\psi_{\alpha,0}(0) \rightarrow \frac{1}{2}$  as  $\alpha \rightarrow 1+$ .  $\square$

## 4 Proof of Theorem 1.4

**Proof.** The reciprocal of the polynomial kernel  $\widehat{K}_N(x, x; w)^{-1}$  (recall its definition in (1.11)) is known in the theory of orthogonal polynomials as the Christoffel function. It satisfies

$$\frac{1}{\widehat{K}_N(x, x; w)} = \min_{\substack{P(x)=1 \\ \deg P \leq N-1}} \int |P(t)|^2 w(t) dt,$$

where the minimum is taken over polynomials  $P$  of degree at most  $N-1$  that take the value 1 at  $x$ . From this extremal property and (1.27) the inequalities

$$\widehat{K}_N(x, x; w_{R,\alpha}^+) \leq \widehat{K}_N(x, x; w_R) \leq \widehat{K}_N(x, x; w_{R,\alpha}^-) \quad (4.1)$$

for  $R > R_\alpha$ , follow.

We assume  $\alpha$  is close enough to 1 so that (1.29) holds. Because of (1.10)-(1.11), and (1.28) we then also have the corresponding behavior for the  $\widehat{K}_N$  kernels

$$\lim_{R \rightarrow \infty} \widehat{K}_N \left( \frac{x}{N}, \frac{y}{N}; w_{R,\alpha}^\pm \right) = \frac{\sin \pi c_\alpha^\pm(x-y)}{\pi(x-y)} \quad (4.2)$$

and in particular if  $y = x$ ,

$$\lim_{R \rightarrow \infty} \frac{1}{N} \widehat{K}_N \left( \frac{x}{N}, \frac{x}{N}; w_{R,\alpha}^\pm \right) = c_\alpha^\pm.$$

By the inequalities (4.1) we get

$$c_\alpha^+ \leq \liminf_{R \rightarrow \infty} \frac{1}{N} \widehat{K}_N \left( \frac{x}{N}, \frac{x}{N}; w_R \right) \leq \limsup_{R \rightarrow \infty} \frac{1}{N} \widehat{K}_N \left( \frac{x}{N}, \frac{x}{N}; w_R \right) \leq c_\alpha^-.$$

The parameter  $\alpha$  can be taken arbitrarily close to 1, and we find because of the assumption (1.30) that

$$\lim_{R \rightarrow \infty} \frac{1}{N} \widehat{K}_N \left( \frac{x}{N}, \frac{x}{N}; w_R \right) = \frac{1}{2} \quad (4.3)$$

uniformly for  $x$  in compacts.

We next use an idea of D.S. Lubinsky [20, p. 919] to estimate  $\widehat{K}_N(x, y)$  for  $x \neq y$  in terms of values on the diagonal. The inequality of Lubinsky is

$$\left| \frac{\widehat{K}_N(x, y; w_{R,\alpha}^+) - \widehat{K}_N(x, y; w_R)}{\widehat{K}_N(x, x; w_R)} \right| \leq \sqrt{\frac{\widehat{K}_N(y, y; w_R)}{\widehat{K}_N(x, x; w_R)}} \sqrt{1 - \frac{\widehat{K}_N(x, x; w_{R,\alpha}^+)}{\widehat{K}_N(x, x; w_R)}}. \quad (4.4)$$

In (4.4) we replace  $x$  and  $y$  by  $x/N$  and  $y/N$ , respectively, and let  $R \rightarrow \infty$ . Then the right-hand side of (4.4) has the limit  $\sqrt{1 - 2c_\alpha^+}$  by (1.29) and (4.3). Applying (4.3) also to the denominator on the left-hand side of (4.4) we obtain

$$\limsup_{R \rightarrow \infty} \left| \frac{1}{N} \widehat{K}_N \left( \frac{x}{N}, \frac{y}{N}; w_{R,\alpha}^+ \right) - \frac{1}{N} \widehat{K}_N \left( \frac{x}{N}, \frac{y}{N}; w_R \right) \right| \leq \frac{\sqrt{1 - 2c_\alpha^+}}{2}. \quad (4.5)$$

From (1.28) and (1.29) we have

$$\lim_{R \rightarrow \infty} \frac{1}{N} \widehat{K}_N \left( \frac{x}{N}, \frac{y}{N}; w_{R,\alpha}^+ \right) = \frac{\sin \pi c_\alpha^+(x - y)}{\pi(x - y)}$$

which when combined with (4.5) leads to

$$\limsup_{R \rightarrow \infty} \left| \frac{\sin \pi c_\alpha^+(x - y)}{\pi(x - y)} - \frac{1}{N} \widehat{K}_N \left( \frac{x}{N}, \frac{y}{N}; w_R \right) \right| \leq \frac{\sqrt{1 - 2c_\alpha^+}}{2}. \quad (4.6)$$

Now let  $\alpha \rightarrow 1$ . Since  $c_\alpha^+ \rightarrow 1/2$  as  $\alpha \rightarrow 1$  we obtain from (4.6)

$$\lim_{R \rightarrow \infty} \frac{1}{N} \widehat{K}_N \left( \frac{x}{N}, \frac{y}{N}; w_R \right) = \frac{\sin \frac{\pi}{2}(x - y)}{\pi(x - y)}. \quad (4.7)$$

From (1.27) and (1.28) we see that  $w_R(x/N) \rightarrow 1$  and  $w_R(y/N) \rightarrow 1$  as  $N \rightarrow \infty$ , and we obtain

$$\lim_{N \rightarrow \infty} \frac{1}{N} K_N \left( \frac{x}{N}, \frac{y}{N}; w_R \right) = \frac{\sin \frac{\pi}{2}(x-y)}{\pi(x-y)}, \quad (4.8)$$

which leads to (1.16) if we replace  $x$  by  $2x$  and  $y$  by  $2y$ .  $\square$

## A Riemann-Hilbert analysis and proof of Proposition 3.2

In this section we prove Proposition 3.2 using the Riemann-Hilbert approach. We apply the Riemann-Hilbert analysis to the weight

$$W_N(x) = (1 - x^2)^{-1/2} e^{-NV_{\alpha,\varepsilon}(x)},$$

that appears in (3.12). The analysis for the weight in (3.13) is similar. The Riemann-Hilbert problem associated with  $W_N$  asks for a  $2 \times 2$  matrix valued function  $Y : \mathbb{C} \setminus [-1, 1] \rightarrow \mathbb{C}^{2 \times 2}$  such that

Y1:  $Y$  is analytic on  $z \in \mathbb{C} \setminus [-1, 1]$  with continuous boundary values  $Y_{\pm}$  on  $(-1, 1)$ ;

Y2:  $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & W_N(x) \\ 0 & 1 \end{pmatrix}$  for every  $x \in (-1, 1)$ ;

Y3:  $Y(z) = (I + O(\frac{1}{z})) \begin{pmatrix} z^N & 0 \\ 0 & z^{-N} \end{pmatrix}$  as  $z \rightarrow \infty$ ;

Y4:  $Y(z) = O \begin{pmatrix} 1 & |z \pm 1|^{-1/2} \\ 1 & |z \pm 1|^{-1/2} \end{pmatrix}$  as  $z \rightarrow \pm 1$ .

The Riemann-Hilbert problem has a unique solution, where  $Y_{11}$  is the monic orthogonal polynomial of degree  $N$  for the weight  $W_N$ . The kernel  $K_N$  is given in terms of  $Y$  as

$$K_N(x, y; W_N) = \frac{\sqrt{W_N(x)W_N(y)}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_+(y)^{-1} Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (A.1)$$

for  $x, y \in (-1, 1)$ .

Let  $\psi_{\alpha,\varepsilon}$  be the density of the equilibrium measure as in (3.7), and let  $\ell_{\alpha}$  be the constant in the variational condition (3.3) for the external field  $V_{\alpha,\varepsilon}$ .

The constant  $\ell_\alpha$  does not depend on  $\varepsilon$ . It can be explicitly calculated, but the precise value is of no interest to us here.

Following [10, 12], we define the  $g$ -function

$$g(z) := \int_{-1}^1 \log(z-t) \psi_{\alpha,\varepsilon}(t) dt, \quad z \in \mathbb{C} \setminus [-\infty, 1],$$

with the principal branch of the logarithm. Then  $T$  defined by

$$T(z) = \begin{pmatrix} e^{-\ell_\alpha} & 0 \\ 0 & e^{\ell_\alpha} \end{pmatrix} Y(z) \begin{pmatrix} e^{-N(g(z)-\ell_\alpha)} & 0 \\ 0 & e^{N(g(z)-\ell_\alpha)} \end{pmatrix} \quad (\text{A.2})$$

satisfies

T1:  $T$  is analytic on  $\mathbb{C} \setminus [-1, 1]$  with continuous boundary values  $T_\pm$  on  $(-1, 1)$ ;

T2:  $T_+(x) = T_-(x) J_T(x)$  for  $x \in (-1, 1)$ , where

$$J_T(x) = \begin{pmatrix} e^{-2\pi i N \int_x^1 \psi_{\alpha,\varepsilon}(s) ds} & \frac{1}{\sqrt{1-x^2}} \\ 0 & e^{2\pi i N \int_x^1 \psi_{\alpha,\varepsilon}(s) ds} \end{pmatrix}; \quad (\text{A.3})$$

T3:  $T(z) = I + O\left(\frac{1}{z}\right)$  as  $z \rightarrow \infty$ ;

T4:  $T$  has the same behavior as  $Y$  as  $z \rightarrow \pm 1$ .

Define

$$\begin{aligned} \rho_{\alpha,\varepsilon}(x) &:= 2\pi\sqrt{1-x^2}\psi_{\alpha,\varepsilon}(x) \\ &= \varepsilon_\alpha - \varepsilon x + \frac{2}{\alpha}\sqrt{1-x^2} \arctan\left(\frac{\sqrt{1-x^2}}{\sqrt{\alpha^2-1}}\right), \quad x \in [-1, 1] \end{aligned} \quad (\text{A.4})$$

where we recall that  $\varepsilon_\alpha$  is given by (3.6). Then  $\rho_{\alpha,\varepsilon}$  has an analytic continuation to

$$U_\alpha := \mathbb{C} \setminus ((-\infty, -\alpha] \cup [\alpha, \infty))$$

which contains the interval  $[-1, 1]$  in its interior. We also use  $\rho_{\alpha,\varepsilon}$  for the analytic continuation. Let

$$\varphi(z) = z + (z^2 - 1)^{1/2}, \quad z \in \mathbb{C} \setminus [-1, 1]$$

be the conformal map from the exterior of  $[-1, 1]$  to the exterior of the unit disk. For each  $\tau \in (1, \varphi(\alpha))$ , the curve

$$\Gamma_\tau = \{z \in \mathbb{C} \setminus [-1, 1] \mid \varphi(z) = \tau\}$$



is an ellipse around  $[-1, 1]$  lying in  $U_\alpha$ . It is then clear that there is an  $M$  such that

$$|\rho'_{\alpha,\varepsilon}(z)| \leq M, \quad z \in \text{int}(\Gamma_\tau) \quad (\text{A.5})$$

and  $M$  is independent of  $\varepsilon \in [-\varepsilon_\alpha, \varepsilon_\alpha]$ , which is easily seen from the simple way that  $\rho_{\alpha,\varepsilon}$  depends on  $\varepsilon$ , see (A.4). Also from (A.4)

$$\rho_{\alpha,\varepsilon}(x) \geq \varepsilon_\alpha - |\varepsilon|, \quad x \in [-1, 1]. \quad (\text{A.6})$$

We let

$$\xi(z) = \int_1^z \frac{\rho_{\alpha,\varepsilon}(s)}{(s^2 - 1)^{1/2}} ds, \quad z \in U_\alpha \setminus (-\infty, 1], \quad (\text{A.7})$$

where  $(s^2 - 1)^{1/2}$  is analytic for  $s \in \mathbb{C} \setminus [-1, 1]$  and positive for  $s > 1$ . The contour of integration in (A.7) is the line segment from 1 to  $z$  in the complex plane. It can be checked that  $e^{\xi(z)}$  has an analytic extension to all of  $U_\alpha \setminus [-1, 1]$ . Then for  $x \in (-1, 1)$ ,

$$\xi_+(x) = -\xi_-(x) = 2\pi i \int_x^1 \psi_{\alpha,\varepsilon}(s) ds,$$

and so we can write the jump matrix (A.3) for  $T$  in terms of  $\xi_\pm$ . There is a factorization

$$\begin{aligned} J_T(x) &= \begin{pmatrix} 1 & 0 \\ i(x^2 - 1)_-^{1/2} e^{-N\xi_-(x)} & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{1-x^2}} \\ -\sqrt{1-x^2} & 0 \end{pmatrix} \\ &\quad \times \begin{pmatrix} 1 & 0 \\ -i(x^2 - 1)_+^{1/2} e^{-N\xi_+(x)} & 1 \end{pmatrix}. \end{aligned} \quad (\text{A.8})$$

We take  $\tau > 1$  close to 1 and define  $S$  by

$$S(z) = \begin{cases} T(z), & z \in \text{ext}(\Gamma_\tau), \\ T(z) \begin{pmatrix} 1 & 0 \\ i(z^2 - 1)^{1/2} e^{-N\xi(z)} & 1 \end{pmatrix}, & z \in \text{int}(\Gamma_\tau) \setminus [-1, 1]. \end{cases} \quad (\text{A.9})$$

Then  $S$  satisfies the following Riemann-Hilbert problem.

S1:  $S$  is analytic on  $\mathbb{C} \setminus (\Gamma_\tau \cup [-1, 1])$  with continuous boundary values on  $\Gamma_\tau \cup (-1, 1)$ ;

S2:  $S_+(x) = S_-(x) \begin{pmatrix} 0 & \frac{1}{\sqrt{1-x^2}} \\ -\sqrt{1-x^2} & 0 \end{pmatrix}$  for  $x \in (-1, 1)$ , and

$S_+ = S_- J_S$  on  $\Gamma_\tau$  where

$$\begin{pmatrix} 1 & 0 \\ -i(z^2 - 1)^{1/2} e^{-N\xi(z)} & 1 \end{pmatrix}; \quad (\text{A.10})$$

S3:  $S(z) = I + O\left(\frac{1}{z}\right)$  as  $z \rightarrow \infty$ ;

S4:  $S$  has the same behavior as  $T$  as  $z \rightarrow \pm 1$ .

Let  $z \in \Gamma_\tau$ . There is  $\theta \in [0, 2\pi]$  such that  $\varphi(z) = \tau e^{i\theta}$ . Then by (A.7)

$$\begin{aligned}\xi(z) &= \int_1^z \frac{\rho_{\alpha,\varepsilon}(s) - \rho(\cos \theta)}{(s^2 - 1)^{1/2}} ds + \rho(\cos \theta) \int_1^z \frac{1}{(s^2 - 1)^{1/2}} ds \\ &= \int_1^z \frac{\rho_{\alpha,\varepsilon}(s) - \rho(\cos \theta)}{(s^2 - 1)^{1/2}} ds + \rho(\cos \theta) \log \varphi(z).\end{aligned}\quad (\text{A.11})$$

Note that the integrand in the integral in the right-hand side of (A.11) is purely imaginary for  $s \in [-1, 1]$ . For the real part of  $\xi(z)$  we may then start integrating at any other point in  $[-1, 1]$  instead of 1. We choose  $\cos \theta$  and since  $|\varphi(z)| = \tau$ , we obtain from (A.11)

$$\operatorname{Re} \xi(z) = \operatorname{Re} \left( \int_{\cos \theta}^z \frac{\rho_{\alpha,\varepsilon}(s) - \rho_{\alpha,\varepsilon}(\cos \theta)}{(s^2 - 1)^{1/2}} ds \right) + \rho_{\alpha,\varepsilon}(\cos \theta) \log \tau. \quad (\text{A.12})$$

We make the change of variables  $s = \varphi^{-1}(w) = \frac{1}{2}(w + \frac{1}{w})$ , and then (A.12) leads to

$$\operatorname{Re} \xi(z) = \operatorname{Re} \left( \int_{e^{i\theta}}^{\tau e^{i\theta}} \frac{\rho_{\alpha,\varepsilon}(\frac{1}{2}(w + \frac{1}{w})) - \rho_{\alpha,\varepsilon}(\cos \theta)}{w} dw \right) + \rho_{\alpha,\varepsilon}(\cos \theta) \log \tau. \quad (\text{A.13})$$

The first term in (A.13) is  $\geq -M(\tau - 1)^2$  because of (A.5). The second term is estimated by (A.6) and we find for  $z \in \Gamma_\tau$ ,

$$\operatorname{Re} \xi(z) \geq (\varepsilon_\alpha - |\varepsilon|) \log \tau - M(\tau - 1)^2. \quad (\text{A.14})$$

Thus if  $\varepsilon$  is in a compact subset of  $(-\varepsilon_\alpha, \varepsilon_\alpha)$  then there are  $c > 0$  and  $\tau > 1$  such that for all  $z$  in a neighborhood of  $\Gamma_\tau$ ,

$$\operatorname{Re} \xi(z) \geq c > 0. \quad (\text{A.15})$$

For such a  $\tau > 1$ , the jump matrix on  $\Gamma_\tau$  in the Riemann-Hilbert problem for  $S$  tends to the identity matrix at an exponential rate as  $N \rightarrow \infty$ .

Define now

$$M(z) = \begin{pmatrix} 1 & -\frac{1}{i(z^2 - 1)^{1/2}} \\ \frac{z - (z^2 - 1)^{1/2}}{2i} & \frac{z + (z^2 - 1)^{1/2}}{2(z^2 - 1)^{1/2}} \end{pmatrix}, \quad z \in \mathbb{C} \setminus [-1, 1]. \quad (\text{A.16})$$

Then it is easy to verify that

M1:  $M$  is analytic on  $\mathbb{C} \setminus [-1, 1]$  with continuous boundary values on  $(-1, 1)$ ;

M2:  $M$  has the same jump as  $S$  on  $(-1, 1)$ ;

M3:  $M(z) = I + O(\frac{1}{z})$  as  $z \rightarrow \infty$ ;

M4:  $M(z)$  has the same behavior as  $S(z)$  as  $z \rightarrow \pm 1$ .

We now introduce the final transformation

$$R(z) = S(z)M^{-1}(z), \quad z \in \mathbb{C} \setminus (\Gamma_\tau \cup [-1, 1]). \quad (\text{A.17})$$

Then  $R$  has analytic continuation across  $[-1, 1]$ , and  $R$  satisfies the following Riemann-Hilbert problem.

R1:  $R$  is analytic on  $\mathbb{C} \setminus \Gamma_\tau$ ;

R2:  $R_+ = R_- J_R$  on  $\Gamma_\tau$ , where  $J_R = M J_S M^{-1}$ , see (A.10);

R3:  $R(z) = I + O(\frac{1}{z})$  as  $z \rightarrow \infty$ .

The jump matrix  $J_R$  satisfies because of (A.10) and (A.15),

$$\begin{aligned} J_R(z) &= M(z) \begin{pmatrix} 1 & 0 \\ -i(z^2 - 1)^{1/2} e^{-N\xi(z)} & 1 \end{pmatrix} M^{-1}(z) \\ &= I + O(e^{-cN}), \quad \text{as } N \rightarrow \infty, \end{aligned} \quad (\text{A.18})$$

uniformly for  $\varepsilon$  in a compact subset of  $(-\varepsilon_\alpha, \varepsilon_\alpha)$ .

Then by standard estimates (see e.g. [1, Sec. 5.9] and [17, Thm. 3.1]) we have that

$$R(z) = I + O(e^{-cN}) \quad \text{as } N \rightarrow \infty, \quad (\text{A.19})$$

uniformly for  $z \in \mathbb{C} \setminus \Gamma_\tau$  and uniformly for  $\varepsilon$  in compact subset of  $(-\varepsilon_\alpha, \varepsilon_\alpha)$ .

To obtain (3.12) we then follow the effect of the transformations  $Y \mapsto T \mapsto S \mapsto R$  on the expression (A.1) for the kernel  $K_N$ . The computations are similar to what is done in [9, 10], and in particular in [2, section 2.7]. Since (A.19) is uniform in  $\varepsilon$  the  $O$ -term in (3.12) is uniform for  $\varepsilon$  in compact subsets of  $(-\varepsilon_\alpha, \varepsilon_\alpha)$  as well.

**Remark** The proof for (3.13) follows along the same lines. Instead of  $\frac{1}{\sqrt{1-x^2}}$  one has  $\sqrt{1-x^2}$  in the 12-entry of the jump matrix  $J_T$  in (A.3). The transformations are similar, but slightly different, and they ultimately lead to an expression for the jump matrix  $J_R$  as

$$J_R(z) = M(z) \begin{pmatrix} 1 & 0 \\ i(z^2 - 1)^{-1/2} e^{-N\xi(z)} & 1 \end{pmatrix} M^{-1}(z),$$

compare with (A.18), with a different  $M$ . Again  $J_R(z) = I + O(e^{-cN})$  for  $z \in \Gamma_\tau$ , and the rest of the argument is the same.

**Remark** If the square root factors  $(1 - x^2)^{\pm 1/2}$  are missing from the weight, then the Riemann-Hilbert analysis becomes more complicated. Then the 12-entry of the jump matrix  $J_T$  is 1, and there is a factorization

$$J_T = \begin{pmatrix} 1 & 0 \\ e^{-N\xi_-} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-N\xi_+} & 1 \end{pmatrix}, \quad (\text{A.20})$$

compare with (A.8), and the jump for  $T$  is written as

$$\left( T \begin{pmatrix} 1 & 0 \\ -e^{-N\xi} & 1 \end{pmatrix} \right)_+ = \left( T \begin{pmatrix} 1 & 0 \\ e^{-N\xi} & 1 \end{pmatrix} \right)_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This suggests to define  $S = T \begin{pmatrix} 1 & 0 \\ -e^{-N\xi} & 1 \end{pmatrix}$  in the region inside  $\Gamma_\tau$  in the upper half plane, and  $S = T \begin{pmatrix} 1 & 0 \\ e^{-N\xi} & 1 \end{pmatrix}$  in the region inside  $\Gamma_\tau$  in the lower half plane. However, then  $S$  has a jump not only on  $\Gamma_\tau \cup [-1, 1]$ , but also on the two intervals  $[-\alpha, -1]$  and  $[1, \alpha]$  and a further analysis is necessary to handle these jumps.

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